Derive eqn 5.10.5 p 210. $\boxed{5-10-9.}$

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<u>Solution</u>: Given that the error is the mean square deviation

$$E \equiv \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)\, dx . \qquad (1) \quad (5.10.4)$$

Since this error is a function of the coefficients $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_M$ we can minimize a function of M variables using the 1st-derivative condition.

That is, the first partial derivative w/ respect to each α_i is zero:

Set $$\frac{\partial E}{\partial \alpha_i} = 0 \quad , i = 1, 2, 3 \ldots M \qquad (2)$$

We will calculate each partial derivative and set it equal to zero to get:

$$\frac{\partial E}{\partial \alpha_i} = 0 = -2\int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \phi_i(x)\, \sigma(x)\, dx \qquad (5.10.5)\ (3)$$

Here's how you do it:

$$E = \int_a^b \sigma(x) \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 dx = \qquad (4)$$

$$= \int_a^b \left[f(x)^2 - 2 f(x) \sum_{n=1}^{M} \alpha_n \phi_n(x) + \sum_{n=1}^{M} \alpha_n \phi_n(x) \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \sigma(x)\, dx \qquad (5)$$

$$= \int_a^b f(x)^2\, \sigma(x)\, dx - 2\int_a^b f(x) \left[\sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \sigma(x)\, dx + \int_a^b \left[\sum_{n=1}^{M} \alpha_n \phi_n(x) \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \sigma(x)\, dx$$

$$\frac{\partial E}{\partial \alpha_i} = 0 - 2\int_a^b f \frac{\partial}{\partial \alpha_i} \left[\sum_{n=1}^{M} \alpha_n \phi_n \right] \sigma(x)\, dx + \int_a^b \sigma(x) \left[\frac{\partial}{\partial \alpha_i} \left(\sum_{n=1}^{M} \alpha_n \phi_n \sum_{n=1}^{M} \alpha_n \phi_n \right) \right] dx$$

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$$\frac{\partial E}{\partial \alpha_i} = -2\int_a^b f(x) \frac{\partial}{\partial \alpha_i}\left[\sum_{n=1}^{M} \alpha_n \phi_n(x)\right] \sigma(x)\, dx + \int_a^b \sigma(x) \cancel{dx} \left[\frac{\partial}{\partial \alpha_i}\left(\sum_{n=1}^{M} \alpha_n \phi_n(x) \sum_{n=1}^{M} \alpha_n \phi_n(x)\right)\right] dx \quad (8)$$

$$= 2\int_a^b f(x)\, \sigma(x) \cancel{dx} \frac{\partial}{\partial \alpha_i}\left[\sum_{n=1}^{M} \alpha_n \phi_n(x)\right] dx + \int_a^b \sigma(x) \cancel{dx} \left[\frac{\partial}{\partial \alpha_i}\left(\sum_{n=1}^{M} \alpha_n \phi_n(x)\right)\sum_{n=1}^{M} \alpha_n \phi_n(x) + \sum_{n=1}^{M} \alpha_n \phi_n(x) \frac{\partial}{\partial \alpha_i}\left(\sum_{n=1}^{M} \alpha_n \phi_n(x)\right)\right]$$

$$= -2\int_a^b f(x)\, \sigma(x) \cancel{dx} \frac{\partial}{\partial \alpha_i}\left[\sum_{n=1}^{M} \alpha_n \phi_n(x)\right] dx + 2\int_a^b \sigma(x) \cancel{dx} \left[\frac{\partial}{\partial \alpha_i}\left(\sum_{n=1}^{M} \alpha_n \phi_n(x)\right)\right] \sum_{n=1}^{M} \alpha_n \phi_n(x)\, dx$$

$$= -2\int_a^b f(x)\, \sigma(x) \frac{\partial}{\partial \alpha_i}\left[\sum_{n=1}^{M} \alpha_n \phi_n(x)\right] dx + 2\int \sigma(x) \left[\frac{\partial}{\partial \alpha_i}\left(\sum_{n=1}^{M} \alpha_n \phi_n(x)\right)\right] \sum_{n=1}^{M} \alpha_n \phi_n(x)\, dx \quad (11$$

$$= -2\int_a^b \left[f(x)\, \sigma(x) - \sigma(x) \sum_{n=1}^{M} \alpha_n \phi_n(x)\right] \frac{\partial}{\partial \alpha_i} \sum_{n=1}^{M} \alpha_n \phi_n(x)\, dx \quad (12)$$

$$\text{Let } \phi_i(x) = \frac{\partial}{\partial \alpha_i} \sum_{n=1}^{M} \alpha_n \phi_n(x) \quad (13)$$

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$$\left[\frac{\partial E}{\partial \alpha_i} = -2 \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \phi_i(x)\, \sigma(x)\, dx \right] \quad (14)$$

(5.10.5)

Done.

Now let's continue to minimize the error E.

$$\frac{\partial E}{\partial \alpha_i} = 0 = -2 \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right] \phi_i(x)\, \sigma(x)\, dx$$

$$0 = \int_a^b f(x)\, \phi_i(x)\, \sigma(x)\, dx - \int_a^b \sum_{n=1}^{M} \alpha_n \phi_n(x)\, \phi_i(x)\, \sigma(x)\, dx \quad (16)$$

Using ~~that~~ the fact that the eigen functions are orthogonal $i=n$ is the only valid state.

So we can drop the $\sum_{n=1}^{M}$ in (16)

$$\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx = \int_a^b \alpha_i\,\phi_i(x)\,\phi_i(x)\,\sigma(x)\,dx$$

$$\alpha_i = \frac{\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx}{\int_a^b \phi_i^2(x)\,\sigma(x)\,dx}$$

$\alpha_i = a_i = d_n$ as before.

This is the α_n that minimizes the error in (1) $\equiv$ 5.10.4 p 210.

Ω

Reproof and show that $\boxed{5-10-10}$

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there is an error in the proof of the minimum error E on page 211.

Solution: To proof that the error E is actually minimized in (1), we will not be using partial derivatives as in Problem 5-10-9. Instead, this derivation proceeds by expanding the square deviation in (1):

$$E = \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)\, dx \qquad (1)$$

$$= \int_a^b \left[f_{(x)}^2 - 2\sum_{n=1}^{M} \alpha_n f(x) \phi_n(x) + \sum_{n=1}^{M} \sum_{l=1}^{M} \alpha_n \alpha_l \phi_n(x) \phi_l(x) \right] \sigma(x)\, dx \qquad (2)$$

Due to the orthogonality of the eigen functions $n = l$ (only survives). Thus, we get

$$E = \int_a^b \left[\sum_{n=1}^{M} \alpha_n^2 \phi_n^2(x) - 2 \sum_{n=1}^{M} \alpha_n f(x) \phi_n(x) + f^2(x) \right] \sigma(x)\, dx \qquad (3)$$

$$= \sum_{n=1}^{M} \int_a^b \left[\alpha_n^2 \phi_n^2 - 2\alpha_n f(x) \phi_n(x) \right] \sigma(x)\, dx + \int_a^b f^2(x)\, \sigma(x)\, dx \qquad (4)$$

$$= \sum_{n=1}^{M} \left[\alpha_n^2 \int_a^b \phi_n^2(x)\, \sigma(x)\, dx - 2\alpha_n \int_a^b f(x) \phi_n(x)\, \sigma(x)\, dx \right] + \int_a^b f^2(x)\, \sigma(x)\, dx$$

Since each α_n appears quadratically, we can get a quadratic in terms of α_n by completing

$$= \sum_{n=1}^{M} \int_a^b \phi_n^2(x)\, \sigma(x)\, dx \left[\alpha_n^2 - 2\alpha_n \frac{\int_a^b f(x) \phi_n(x)\, \sigma(x)\, dx}{\int_a^b \phi_n^2(x)\, \sigma(x)\, dx} \right] + \int_a^b f^2(x)\, \sigma(x)\, dx \qquad (6)$$

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$$E=\sum_{n=1}^{M}\int_a^b \phi_n^2(x)\,\sigma(x)\,dx\left[\alpha_n^2-2\alpha_n\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}+\left(\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}\right)^2\right. + \tag{7}$$

$$\left.-\left(\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}\right)^2\right]+\int_a^b f^2(x)\,\sigma(x)\,dx$$

$$E=\sum_{n=1}^{M}\int_a^b \phi_n^2(x)\,\sigma(x)\,dx\left[\underbrace{\left(\alpha_n-\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}\right)^2}-\left(\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx}\right)^2\right]+\int_a^b f^2\sigma\,dx \tag{8}$$

To minimize E set this term $=0$ to get

$$\alpha_n=\frac{\int_a^b f(x)\phi_n(x)\sigma(x)\,dx}{\int_a^b \phi_n^2 \ldots} \tag{9}$$

this is the same

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And

$$E_{min} \equiv \sum_{n=1}^{M} \int_a^b \phi_n^2(x)\, \sigma(x)\, dx \left[- \left(\frac{\int_a^b f(x)\phi_n(x)\,\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\,\sigma(x)\,dx} \right)^2 \right] + \int_a^b f^2(x)\,\sigma(x)\,dx \qquad (10)$$

$$= \sum_{n=1}^{M} \int_a^b \phi_n^2(x)\,\sigma(x)\,dx \left[-\alpha_n^2 \right] + \int_a^b f^2(x)\,\sigma(x)\,dx$$

$$\left[E_{min} = \int_a^b f^2(x)\,\sigma(x)\,dx - \sum_{n=1}^{M} \alpha_n^2 \int \phi_n^2(x)\,\sigma(x)\,dx \right] \qquad (12)$$

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