For the equation

$$z''+4z'+3z = 130e^{2it}$$

Show that the trial solution $z = Ae^{2it}$ leads to ***(-1 + 8i)A = 130***. Solve for ***A*** and thus obtain

$$z = (-2 - 16i)(\cos 2t + i\sin 2t)$$

Solution:

$$\text{Since} \quad z''+4z'+3z = 130e^{2it} \quad \text{and} \quad \begin{bmatrix} z = Ae^{2it} \\ z' = 2iAe^{2it} \\ z'' = 4i^2Ae^{2it} \\ \quad = -4Ae^{2it} \end{bmatrix}$$

$$\therefore \qquad z''+4z'+3z = 130e^{2it}$$

$$-4Ae^{2it} + 8iAe^{2it} + 3Ae^{2it} = 130e^{2it}$$

$$-4A + 3A + 8iA = 130$$

$$\begin{bmatrix} (-1 + 8i)A = 130 \end{bmatrix}$$

$$A = \frac{130}{(-1 + 8i)} \cdot \frac{(-1 - 8i)}{(-1 - 8i)} = \frac{130(-1 - 8i)}{1 + 8i - 8i - 64i^2}$$

$$\begin{bmatrix} A = \frac{130(-1 - 8i)}{65} = 2(-1 - 8i) = -2 - 16i \end{bmatrix}$$

Therefore,

$$z = Ae^{2it} = (-2 - 16i)e^{2it}$$

$$z = (-2 - 16i)[\cos 2t + i\sin 2t]$$

(B)

Using the result in (A) obtain particular solutions of

$$x''+4x'+3x = 130\cos 2t$$

$$y''+4y'+3y = 130\sin 2t$$

Solution:

$$Since \quad \begin{bmatrix} \cos 2t = \operatorname{Re} e^{2it} \to x \\ \sin 2t = \operatorname{Im} e^{2it} \to y \end{bmatrix} ,$$

let's solve instead,

$$z''+4z'+3z = 130e^{2it}$$

$$Where$$

$$z = (-2 - 16i)(\cos 2t + i\sin 2t)$$

$$= -2\cos 2t - 2i\sin 2t - 16i\cos 2t - 16i^2 \sin 2t$$

$$= -2\cos 2t - (2\sin 2t + 16\cos 2t)i + 16\sin 2t$$

$$z = -2\cos 2t + 16\sin 2t - (2\sin 2t + 16\cos 2t)i$$

$$\therefore$$

$$\begin{bmatrix} x(t) = -2\cos 2t + 16\sin 2t \\ y(t) = -2\sin 2t - 16\cos 2t \end{bmatrix}$$

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By use of an equivalent complex equation or by a trial solution of the form ***Asin(st) + Bcos(st)*** , as you prefer, obtain particular solutions of:

$$x''-3x'+2x = 20\cos 2t$$

$$y''-3y'+2y = 20\sin 2t$$

(A)

Solution:

$$\text{Since} \quad \begin{bmatrix} \cos 2t = \operatorname{Re} e^{2it} \rightarrow x \\ \sin 2t = \operatorname{Im} e^{2it} \rightarrow y \end{bmatrix}$$

So consider instead

$$\text{Since} \quad z''-3z'+2z = 20e^{2it} \quad \text{and} \quad \begin{bmatrix} z = Ae^{2it} \quad \text{trial Solution} \\ z' = 2iAe^{2it} \\ z'' = 4i^2 Ae^{2it} \\ \quad = -4Ae^{2it} \end{bmatrix}$$

$$\therefore \qquad z''-3z'+2z = 20e^{2it}$$

$$-4Ae^{2it} - 6iAe^{2it} + 2Ae^{2it} = 20e^{2it}$$

$$-2A - 3iA + A = -A - 3iA = 10$$

$$\begin{bmatrix} (-1 - 3i)A = 10 \end{bmatrix}$$

$$A = \frac{10}{(-1-3i)} \cdot \frac{(-1+3i)}{(-1+3i)} = \frac{10(-1+3i)}{1 - 3i + 3i - 9i^2}$$

$$\begin{bmatrix} A = \frac{10(-1+3i)}{10} = -1 + 3i \end{bmatrix}$$

$$z = Ae^{2it} = (-1+3i)e^{2it}$$

$$= (-1+3i)(\cos 2t + i \sin 2t)$$

$$= -\cos 2t - i \sin 2t + 3i \cos 2t + 3i^2 \sin 2t$$

$$z = -\cos 2t - 3 \sin 2t + (-\sin 2t + 3 \cos 2t)i$$

Thus,

$$\begin{bmatrix} x_p = -\cos 2t - 3\sin 2t \\ y_p = -\sin 2t + 3\cos 2t \end{bmatrix}$$ (21) (22)

(B)

Now try this one:

$$x'' + x' + 17x = 17\cos 4t$$
$$y'' + y' + 17y = 17\sin 4t$$

Solution:

$$Since \quad \begin{bmatrix} \cos 4t = \text{Re}\, e^{4it} \\ \sin 4t = \text{Im}\, e^{4it} \end{bmatrix}$$

So consider instead

$$Since \quad z'' + z' + 17z = 17e^{4it} \quad and \quad \begin{bmatrix} z = Ae^{4it} \quad trial\ Solution \\ z' = 4iAe^{4it} \\ z'' = 16i^2 Ae^{4it} \\ = -16Ae^{4it} \end{bmatrix}$$

$$\therefore \qquad z'' + z' + 17z = 17e^{4it}$$

$$-16Ae^{2it} + 4iAe^{2it} + 17Ae^{2it} = 17e^{4it}$$

$$-16A + 4iA + 17A = A + 4iA = 17$$

$$\begin{bmatrix} (1 + 4i)A = 17 \end{bmatrix}$$

$$A = \frac{17}{(1+4i)} \cdot \frac{(1-4i)}{(1-4i)} = \frac{17(1-4i)}{1 - 4i + 4i - 16i^2}$$

$$\begin{bmatrix} A = \frac{17(1-4i)}{17} = 1 - 4i \end{bmatrix}$$

So,

$$z = Ae^{4it} = (1 - 4i)e^{4it} = (1 - 4i)[\cos 4t + i \sin 4t]$$

$$= \cos 4t + i \sin 4t - 4i \cos 4t - 4i^2 \sin 4t$$

$$z = \cos 4t + 4 \sin 4t + (\sin 4t - 4 \cos 4t)i$$

$\therefore$

$$\begin{bmatrix} x_p = \cos 4t + 4 \sin 4t \\ y_p = \sin 4t - 4 \cos 4t \end{bmatrix}$$

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Obtain a particular solution of:

$$x''+2x'+5x = 20e^t \cos 2t$$

$$y''+2y'+5y = 20e^t \sin 2t$$

Solution:

Since ***cos(2t)*** and ***sin(2t)*** are the ℝeal and Imaginary parts of e^{2it} and

$$\cos 2t = \text{Re } e^{2it} \qquad and \qquad \sin 2t = \text{Im } e^{2it}$$

let’s consider instead

$$z''+2z'+5z = 20e^t e^{2it} = 20e^{2it+t} = 20e^{(1+2i)t} .$$

A trial solution of the form

$$z = Ae^{(1+2i)t} \quad ; \quad z' = (1 + 2i)Ae^{(1+2i)t}$$

$$z'' = (1 + 2i)^2 Ae^{(1+2i)t} = (1 + 2i)(1 + 2i)Ae^{(1+2i)t}$$

$$z'' = [1 + 2i + 2i + 4i^2]Ae^{(1+2i)t} = (-3 + 4i)Ae^{(1+2i)t}$$

$$\therefore \qquad z''+2z'+5z = 20e^{(1+2i)t}$$

$$(-3 + 4i)Ae^{(1+2i)t} + 2(1 + 2i)Ae^{(1+2i)t} + 5Ae^{(1+2i)t} =$$

$$A(-3 + 4i + 2 + 4i + 5) = 20$$

$$A(4 + 8i) = 20$$

$$A(1 + 2i) = 5$$

$$A = \frac{5}{(1 + 2i)} \cdot \frac{(1 - 2i)}{(1 - 2i)} = \frac{(1 - 2i)}{1 - 2i + 2i - 4i^2}$$

$$\left[\quad A = \frac{5(1 - 2i)}{5} = (1 - 2i) \quad \right]$$

$$z = (1 - 2i)e^{(1+2i)t} = (1 - 2i)e^{t}e^{2it}$$

$$z = (1 - 2i)e^{t}(\cos 2t + i \sin 2t)$$

Thus

$$z = e^{t}[\cos 2t + i \sin 2t - 2i \cos 2t - 2i^2 \sin 2t]$$

$$= e^{t}[\cos 2t + 2 \sin 2t + (\sin 2t - 2 \cos 2t)i]$$

$$\therefore$$

$$\left[\begin{array}{l} x = \operatorname{Re} z = e^{t}[\cos 2t + 2 \sin 2t] \\ y = \operatorname{Im} z = e^{t}(\sin 2t - 2 \cos 2t) \end{array} \right]$$

Method 2. (31)

We can also solve each equation separately

$$20e^{t} \sin 2t = y''+2y'+5y$$

$$= (D^2 + 2D + 5)y$$

$$Let \qquad y = ue^{t}$$

$$20e^{t} \sin 2t = (D^2 + 2D + 5)y$$

$$\begin{aligned} 20e^t \sin 2t &= (D^2 + 2D + 5)ue^t \\ &= e^t((D+1)^2 + 2(D+1) + 5)u \\ &= e^t(D^2 + 2D + 1 + 2D + 2 + 5)u \\ &= e^t(D^2 + 4D + 8)u \end{aligned}$$

$$20 \sin 2t = (D^2 + 4D + 8)u$$

$$(D^2 + 4D + 8)u = 20 \sin 2t$$

Here is where we can use a complex or a non-complex implant

e^{it} *or* $A \cos bt + B \sin bt$ respectively.

$$Let \quad u = A \cos 2t + B \sin 2t \qquad @ \; b = 2$$

$$u' = -2A \sin 2t + 2B \cos 2t$$

$$u'' = -4A \cos 2t - 4B \sin 2t$$

$\therefore$

$$\begin{aligned} 20 \sin 2t &= (D^2 + 4D + 8)u \\ &= -4A \cos 2t - 4B \sin 2t - 8A \sin 2t + \\ &\quad + 8B \cos 2t + 8A \cos 2t + 8B \sin 2t \end{aligned}$$

$$20 \sin 2t = (4A + 8B) \cos 2t + (4B - 8A) \sin 2t \qquad (60)$$

$$Where \begin{bmatrix} 4A + 8B = 0 \\ A + 2B = 0 \\ A = -2B \end{bmatrix} and \begin{bmatrix} 8B - 8A = 20 \\ B - 2A = 5 \\ B - 2(-2B) = B + 4B = 5 \end{bmatrix}$$

$$So \quad B = 1 \quad \cup \quad A = -2B = -2$$

$$\therefore \quad u = -2 \cos 2t + \sin 2t$$

$$y = ue^t = e^t[-2 \cos 2t + \sin 2t] \qquad (72)$$

The same as in (31).

To find ***x(t)*** set equation(60) = ***20 cos(2t)*** **.**

$\therefore$

$$20\cos 2t = (4A + 8B)\cos 2t + (4B - 8A)\sin 2t$$

$$Where \begin{bmatrix} 4B - 8A = 0 \\ B - 2A = 0 \\ B = 2A \end{bmatrix} \quad and \quad \begin{bmatrix} 4A + 8B = 20 \\ A + 2B = 5 \\ A + 4A = 5 \end{bmatrix}$$

$$A = 1 \quad \cup \quad B = 2$$

$$\therefore \quad u = A\cos 2t + B\sin 2t$$

$$u = \cos 2t + 2\sin 2t$$

$$x(t) = ue^t = e^t(\cos 2t + 2\sin 2t)$$

Same answer as in equation (30)

(B)

Obtain a particular solution of:

$$x'' - x' - 6x = 102e^{3t}\cos 3t$$

$$y'' - y' - 6x = 102e^{3t}\sin 3t$$

Solution: Method 1

Since ***cos(3t)*** and ***sin(3t)*** are the ℝeal and Imaginary parts of e^{3it} and

$$\cos 3t = \operatorname{Re} e^{3it} \qquad and \qquad \sin 3t = \operatorname{Im} e^{3it}$$

let’s consider instead

$$z'' - z' - 6z = 102e^{3t}e^{3it} = 102e^{3it+3t} = 102e^{(3+3i)t} \quad .$$

A trial solution of the form

$$z = Ae^{(3+3i)t} \quad ; \quad z' = (3 + 3i)Ae^{(3+3i)t}$$

$$z'' = (3 + 3i)^2 Ae^{(3+3i)t} = (3 + 3i)(3 + 3i)Ae^{(3+3i)t}$$

$$z'' = [9 + 18i + 9i^2]Ae^{(3+3i)t} = 18iAe^{(3+3i)t}$$

$$\therefore \qquad z''-z'-6z = 102e^{(3+3i)t}$$

$$18iAe^{(3+3i)t} - (3+3i)Ae^{(3+3i)t} - 6Ae^{(3+3i)t} =$$

$$A(18i - 3i - 3i - 6) = 102$$

$$A(-9 + 15i) = 102$$

$$3A(-3 + 5i) = 102$$

$$A = \frac{34}{(-3+5i)} \cdot \frac{(-3-5i)}{(-3-5i)} = \frac{34(-3-5i)}{9 + 15i - 15i - 25i^2}$$

$$\left[\quad A = \frac{34(-3-5i)}{9+25} = \frac{34(-3-5i)}{34} = -3 - 5i \quad \right]$$

$$z = Ae^{(3+3i)t} = (-3-5i)e^{(3+3i)t} = (-3-5i)e^{3t}e^{3it}$$

$$z = (-3-5i)e^{3t}(\cos 3t + i \sin 3t)$$

Thus

$$z = e^{3t}[-3\cos 3t - 3i \sin 3t - 5i \cos 3t - 5i^2 \sin 2t]$$

$$= e^{t}[-3\cos 3t + 5 \sin 3t + (-3\sin 3t - 5\cos 3t)i \qquad (102)$$

$$\therefore$$

$$\left[\begin{array}{l} x = \operatorname{Re} z = e^{t}[-3\cos 3t + 5\sin 3t] \\ y = \operatorname{Im} z = e^{t}(-3\sin 3t - 5\cos 3t) \end{array} \right] \qquad (104)$$

Method 2.

We can also solve each equation separately

$$102e^{3t} \sin 3t = y''-y'-6y$$

$$= (D^2 - D - 6)y$$

Let $y = ue^{3t}$ *and use the slide THM.*

$$102e^{3t} \sin 3t = (D^2 - D - 6)y$$

$$102e^{3t}\sin 3t = (D^2 - D - 6)ue^{3t}$$
$$= e^{3t}((D+3)^2 - (D+3) - 6)u$$
$$= e^{3t}(D^2 + 6D + 9 - D - 3 - 6)u$$
$$= e^{3t}(D^2 + 6D - D)u$$
$$102\sin 3t = (D^2 + 5D)u$$
$$(D^2 + 5D)u = 102\sin 3t$$

Here is where we can use a complex or a non-complex implant

e^{3it} *or* $A\cos bt + B\sin bt$ respectively.

$$Let \quad u = A\cos 3t + B\sin 3t \qquad @\ b = 3$$
$$u' = -3A\sin 3t + 3B\cos 3t$$
$$u'' = -9A\cos 3t - 9B\sin 3t$$
$$\therefore$$
$$102\sin 3t = (D^2 + 5D)u$$
$$= -9A\cos 3t - 9B\sin 3t - 15A\sin 3t + 15B\cos 3t$$
$$102\sin 3t = (-9A + 15B)\cos 3t + (-9B - 15A)\sin 3t \qquad (118)$$

$$Where \begin{bmatrix} -9A + 15B = 0 \\ 9A = 15B \\ A = \frac{15}{9}B = \frac{5}{3}B \end{bmatrix} and \begin{bmatrix} -9B - 15A = 102 \\ -9B - (15)\frac{5}{2}B = \\ -9B - 25B = -34B = 102 \\ B = \frac{-102}{34} = -3 \end{bmatrix}$$

Therefore, ***B = -3 and A = (5/3)B = -5***

$$\therefore \quad u = -5\cos 3t - 3\sin 3t$$

$$y = ue^{3t} = e^{t}[-5\cos 3t - 3\sin 3t] \qquad (122)$$

$$y = e^{3t}[-3\sin 3t - 5\cos 3t]$$

The same as in (104).

To find ***x(t)*** set equation(118) = ***102 cos(3t)*** **.** And,
$\therefore$

$$102\cos 3t = (-9A + 15B)\cos 3t + (-9B - 15A)\sin 3t$$

$$\text{Where} \quad \begin{bmatrix} -9B - 15A = 0 \\ 15A = -9B \\ A = \frac{-9}{15}B = \frac{-3}{5}B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -9A + 15B = 102 \\ -9A + (15)\frac{-5}{3}A = 102 \\ -9A - 25A = 102 \\ -34A = 102 \\ A = \frac{-102}{34} = -3 \end{bmatrix}$$

$$A = -3 \quad \cup \quad B = \frac{-5}{3}A = 5$$

$$\therefore \quad u = A\cos 3t + B\sin 3t$$

$$u = -3\cos 3t + 5\sin 3t$$

$$x(t) = ue^{3t} = e^{3t}(-3\cos 3t + 5\sin 3t)$$

Same answer as in equation (104)

Solve $z' = e^{(a+ib)t}$ (1)

Separate into ℝeal and Imaginary parts, obtain formula (2) below,

$$\int e^{at}\cos(bt)dt = e^{at}\left[\frac{a\cos bt + b\sin bt}{a^2 + b^2}\right] + c \quad (2)$$

and then solve formula (3),

$$\int e^{at}\sin(bt)dt \quad (3)$$

(A)

Solution:

$$z' = e^{(a+ib)t} = e^{at}e^{ibt} = e^{at}[\cos bt + i\sin bt]$$

$$\text{Re } z' = e^{at}\cos bt \quad \cup \quad \text{Im } z' = e^{at}\sin bt$$

$$\therefore \quad \text{Re } z' = \text{Re}\frac{dz}{dt} = e^{at}\cos bt$$

$$\text{Re } z = \int \text{Re } z'\, dz = \int e^{at}\cos bt\, dt$$

Let

$$\int e^{at}\cos bt\, dt = \int u dv = uv - \int v du$$

$$\textit{If} \quad \begin{bmatrix} u = e^{at} \\ du = ae^{at}dt \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} dv = \cos(bt)\, dt \\ v = \int dv \\ v = \int(\cos bt)dt = \frac{1}{b}\sin bt \end{bmatrix}$$

Then,

$$\int e^{at}(\cos bt)dt = uv - \int v\,du$$

$$= \frac{e^{at}}{b}\sin bt - \int \frac{a}{b}e^{at}\sin(bt)dt$$

$$= \frac{e^{at}}{b}\sin bt - \frac{a}{b}\int e^{at}\sin(bt)dt$$

$$Let \quad A = \int e^{at}\sin(bt)dt = \int udv = uv - \int vdu$$

Then let

$$\begin{bmatrix} u = e^{at} \\ du = ae^{at}dt \end{bmatrix} \quad and \quad \begin{bmatrix} dv = (\sin bt)dt \\ v = \int dv = \int \sin(bt)dt \\ v = \frac{-1}{b}\cos bt \end{bmatrix}$$

$$\int e^{at}\sin(bt)dt = uv - \int vdu \qquad (16)$$

$$= -\frac{e^{at}}{b}\cos bt + \frac{a}{b}\int e^{at}\cos(bt)dt$$

Thus,

$$\int e^{at}\cos(bt)dt =$$

$$= \frac{e^{at}}{b}\sin bt - \frac{a}{b}\left[\frac{-e^{at}}{b}\cos bt + \frac{a}{b}\int e^{at}\cos(bt)dt\right]$$

$$\int e^{at} \cos bt \, dt =$$

$$= \frac{e^{at}}{b} \sin bt + \frac{a}{b^2} e^{at} \cos bt - \frac{a^2}{b^2} \int e^{at} \cos bt \, dt$$

$$\int e^{at} \cos bt \, dt + \frac{a^2}{b^2} \int e^{at} \cos bt \, dt =$$

$$= \frac{e^{at}}{b} \sin bt + \frac{a}{b^2} e^{at} \cos bt$$

$$\int e^{at} \cos bt \, dt \left(1 + \frac{a^2}{b^2} \right) =$$

$$\int e^{at} \cos bt \, dt \left(\frac{b^2 + a^2}{b^2} \right) = \frac{e^{at}}{b} \sin bt + \frac{a}{b^2} e^{at} \cos bt$$

$$= e^{at} \left[\frac{b \sin bt + a \cos bt}{b^2} \right]$$

$$\int e^{at} \cos bt \, dt = e^{at} \left[\frac{b \sin bt + a \cos bt}{a^2 + b^2} \right] + c$$

(B)

Here's how to solve the last one.

$$\int e^{at} \sin(bt) dt = -\frac{e^{at}}{b} \cos bt + \frac{a}{b} \int e^{at} \cos(bt) dt$$

$$= -\frac{e^{at}}{b} \cos bt + \frac{a}{b} e^{at} \left[\frac{a \cos bt + b \sin bt}{a^2 + b^2} \right]$$

Let ***w*** be a nonzero ℝeal constant. Solve

$$(D + i\omega)(D - i\omega)z = e^{i\omega t} \qquad (1)$$

and thus show that the equations below have the indicated solutions:

$$x'' + w^2 x = \cos wt \qquad (2)$$

$$x = \frac{t \sin wt}{2w} \qquad (2a)$$

$$y'' + w^2 y = \sin wt \qquad (3)$$

$$y = -\frac{t \cos wt}{2w} \qquad (3a)$$

Solution:

Since

$$\cos wt = \text{Re}\, e^{iwt} \quad \cup \quad \sin wt = \text{Im}\, e^{iwt},$$

let's use instead

$$z'' + w^2 z = e^{iwt} \qquad (6)$$

$$where \quad \begin{bmatrix} \text{Re}\, z = x(t) & eqn(2a) \\ \text{Im}\, z = y(t) & \end{bmatrix}$$

Note:

$$(D + iw)(D - iw)z = e^{iwt}$$

is the same as equation(6). See

$$(D + iw)(D - iw)z = e^{iwt} \qquad (10)$$

$$e^{iwt} = (D + iw)(D - iw)z = (D^2 - i^2 w^2)z$$

$$e^{iwt} = (D^2 + w^2)z = z'' + w^2 z \equiv eqn(6)$$

So we'll go on and solve (10) in its present form $\begin{bmatrix} \text{Re}\, z = x(t) \\ \text{Im}\, z = y(t) \end{bmatrix}$

Let

$$(D + iw)(D - iw)z = e^{iwt} \quad and \quad z = ue^{iwt}$$

Therefore,

$$e^{iwt} = (D + iw)(D - iw)z$$
$$= (D + iw)(D - iw)ue^{iwt}$$
$$= e^{iwt}((D + iw) + iw)((D + iw) - iw)u$$
$$1 = (D + 2iw)Du$$
$$Let \quad v = Du$$
$$\therefore$$
$$1 = (D + 2iw)Du = (D + 2iw)v = Dv + 2iwv$$

v = a constant (See chapter 6.2.129)
So, ***Dv +2iwv = 0 + 2iwv = 1***

$$v = \frac{1}{2iw} = Du = u' = \frac{du}{dt}$$
$$du = \frac{dt}{2iw}$$
$$\int du = \int \frac{dt}{2iw}$$
$$u = \frac{t}{2iw}$$
$$z = ue^{iwt} = \frac{t}{2iw} e^{iwt} = \frac{t}{2iw}[\cos wt + i \sin wt]$$
$$= \frac{t}{2iw}\left(\frac{i}{i}\right)[\cos wt + i \sin wt] = \frac{-it}{2w}[\cos wt + i \sin wt]$$
$$z = \frac{t}{2w}[-i \cos wt - i^2 \sin wt] = \frac{t}{2w}[\sin wt - i \cos wt]$$

Here are solutions to equations {2,3}.

$$\operatorname{Re} z = x(t) = \frac{t \sin wt}{2w}$$
$$\operatorname{Im} z = y(t) = -\frac{t \cos wt}{2w}$$

Chapter 6 Section 5

Existence and Uniqueness

Sometimes when we are attempting to solve problems it is always helpful if not encouraging to now if there is a solution. And still better, knowing whether the solution is unique or just one of many. This section deals with uniqueness theorems. As stated by "Grossman", *Differential Equations*, 1991, chap 4.2, page 80:

① An existence theorem asserts that a problem has at least one solution,

② a uniqueness theorem asserts that a problem has at most one solution.

And in addition that, "...There is nothing wrong with looking for a solution before you know it exists, and if you find it the question of existence is settled then and there."

At the stage you should already know the importance of complex numbers in physics and engineering. The primary theorem of this section is:

Theorem 6.5.1 — Let ***f*** be a continuous ℝeal or complex-valued function or an open interval ***I*** containing the point t_o . Then the following initial value problem has one, and only one, solution:

$$Ty = f \text{ on } I , \quad y(t_o) = y_o \quad , \quad y'(t_o) = y_1 .$$

So, let's begin!

Write two linearly independent ℝeal solutions to the 9 differential equations given below:

(A)

Given: ***y" + 3y' = 54y*** .

Solution:

$$y''+3y' = 54y$$
$$0 = y''+3y'-54y$$
$$= (D^2 + 3D - 54)y$$
$$Let \quad y = e^{\lambda t} = e^{st}$$
$$\therefore$$
$$0 = (D^2 + 3D - 54)y = (D^2 + 3D - 54)e^{\lambda t}$$
$$Use \quad the \quad " slide - rule"$$
$$0 = e^{\lambda t}(\lambda^2 + 3\lambda - 54)$$
$$0 = (\lambda^2 + 3\lambda - 54) \quad Characteristic \quad eqn.$$
$$0 = (\lambda + 9)(\lambda - 6) \quad @ \quad \lambda \in \{-9,6\}$$
$$\therefore \quad y_1 = e^{\lambda_1 t} \quad ; \quad y_2 = e^{\lambda_2 t}$$

Or as a solution vector:

$$y = \bar{\varphi}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{-9t} \\ e^{6t} \end{pmatrix} = (y_1, y_2) = (e^{-9t}, e^{6t})$$

(B)

Given: $y'' + y = 2y'$. (20)

Solution:

$$y''+y = 2y'$$

$$0 = y''-2y'+y$$

$$= (D^2 - 2D + 1)y$$

$$Let \quad y = e^{\lambda t} = e^{st}$$

$$\therefore$$

$$0 = (D^2 - 2D + 1)y = (D^2 - 2D + 1)e^{\lambda t}$$

$$Use \quad the \quad "slide - rule"$$

$$0 = e^{\lambda t}(\lambda^2 - 2\lambda + 1)$$

$$0 = (\lambda^2 - 2\lambda + 1) \quad Characteristic \quad eqn.$$

$$0 = (\lambda - 1)(\lambda - 1) \quad @ \quad \lambda \in \{1,1\}$$

$$\therefore \quad y_1 = e^{\lambda_1 t} = e^t \quad \cup \quad y_2 = e^{\lambda_2 t} = te^{\lambda_1 t} = te^t$$

Note that $\lambda = 1$ is a double root.
Or as a solution vector:

$$\bar{\varphi}(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^t \\ te^t \end{pmatrix} = e^t \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$y = (y_1, y_2) = (e^t, te^t)$$